On the Axiom of Choice and Infinite Graphs

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The De Bruijn-Erdős theorem (1951) states that an infinite graph G has the chromatic number at most k if every finite subgraph of G can be colored by k colors. In the Zermelo-Fraenkel set theory with the axiom of choice (ZFC) many proofs of this theorem are discussed and all proofs of this theorem use a statement which can be derived from the axiom of choice. A consequence of the De Bruijn-Erdős theorem is that any infinite graph can be colored by 4 colors. This sentence and many other statements such as the axiom of choice can be formalized in the second-order logic and, in connection with the Henkin interpretations (HPL), theorems such as Bernstein's equivalence theorem and the fixed-point theorem of Bourbaki are provable, the axiom of choice follows from the well-ordering theorem, Zorn's lemma follows from the axiom of choice, etc. On the other hand, the well-ordering theorem for unary predicates is independent of axiom of choice for binary predicates and of the trichotomy law for unary predicates. Thus the question arises whether the statement that any infinite graph can be colored by a finite number of colors can be derived from the axiom of choice or the trichotomy law in the socond order logic. We answer the question in the negative by providing some counterexamples.

[1] R. Diestel: Graph Theory, Springer (2006).

[2] C. Gaßner: The Axiom of Choice in Second-Order Predicate Logic, Mathematical Logic Quarterly 40 (1994), 533-546.